CHARACTERIZATIONS OF H_v-Γ-SEMIGROUPS THROUGH INTUITIONISTIC FUZZY H_v-IDEALS

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Abstract: As a generalization of fuzzy sets, the notion of intuitionistic fuzzy sets was introduced by Atanassov [6], and applications of intuitionistic fuzzy concepts have already been done by Atanassov and many others in algebra, topological space, knowledge engineering, natural language, and neural network etc. The concept of hyperstructure first was introduced by Marty [33]. Vougiouklis [43], in the fourth AHA congress (1990), introduced the notion of H_v-structures. Recently, well known authors such as Davvaz, Dudek, Jun, Zhan, Cristea etc. have studied and discussed the intuitionistic fuzzification of different kinds of hyperstructures. The notion of intuitionistic fuzzy sets was introduced by Atanassov as a generalization of the notion of fuzzy sets. In this paper, we deal with H_v-Γ-semigroups which is a generalization of Γ-semigroups and H_v-semigroups. Using Atanassov idea, we apply the concept of intuitionistic fuzzy sets to H_v-Γ-semigroups initiating this kind of study. We introduce the notion of an intuitionistic fuzzy H_v-ideal of an H_v-Γ-semigroup and different properties and characterizations of them are investigated and obtained extending some results obtained in H_v-rings. Also some natural equivalence relations on the set of all intuitionistic fuzzy H_v-ideals of an H_v-Γ-semigroup are investigated.
Keywords: Algebraic hyperstructure, H_v-Γ-semigroup, intuitionistic fuzzy H_v-ideal.

1. INTRODUCTION AND PRELIMINARIES

Hyperstructures, as a natural extension of classical algebraic structures, was introduced in 1934, by F. Marty, a French mathematician, at the 8th Congress of Scandinavian Mathematicians [33]. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. A lot of papers and several books have been written on hyperstructure theory, see [10-12], [42].
As it is well known, Vuogiouklis [33] in the fourth AHA congress (1990), introduced the notion of H_v-structures satisfying the weak axioms where the non-empty intersection replaces the equality. The concept of H_v-structures constitutes a generalization of the well-known algebraic hyperstructures (hypergroup, hyperring, hypermodule and so on). The study of H_v-structure theory has been pursued in many directions by Vuogiouklis, Davvaz, Spartalis and others. For definition, results and applications on H_v-structures, see [13-18, 22-24, 29, 30, 37-39, 44-46].
Uncertainty is an attribute of information and uncertain data are presented in various domains and the most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh in his classic paper [47]. Fuzzy sets handle such situations by attributing a degree to which a certain object belongs to a set. The theory of fuzzy sets provides a natural framework for generalizing some of the notions of classical algebraic structures. Fuzzy semigroups have been first considered by Kuroki [31]. After the introduction of the concept of fuzzy sets by Zadeh, several researches conducted the researches on the generalizations of the notions of fuzzy sets with huge applications in computer, logics and many branches of pure and applied mathematics. In 1971, Rosenfeld [36] defined the concept of fuzzy group. Since then many papers have been published in the field of fuzzy algebra. Recently fuzzy set theory has been well developed in the context of hyperalgebraic structure theory. A recent book [11] contains a wealth of applications. In [19], Davvaz introduced the concept of fuzzy hyperideals in a semihypergroup. But in fuzzy sets theory, there is no means to incorporate the hesitation or uncertainty in the membership degrees. As an important generalization of the notion of fuzzy sets on a non-empty set X, in 1983, Atanassov introduced in [5] the concept of intuitionistic fuzzy sets on a non-empty set X which give both a membership degree and a non-membership degree. The only constraint on these two degrees is that the sum must be smaller than or equal to 1. Atanassovs intuitionistic fuzzy sets as a generalization of fuzzy sets can be useful in situations when description of a problem by a (fuzzy) linguistic variable, given in terms of a membership function only, seems too rough. In [20], using Atanassov idea, Davvaz established the intuitionistic fuzzification of the concept of hyperideals in a semihypergroup and investigated some of their properties. Recently in [28], it is studied the structure of semihypergroups through intuitionistic fuzzy sets. Recently, in [4, 26], it is initiated a study on intuitionistic fuzzy sets in Γ-semihypergroups which was introduced and studied recently by Davvaz, Hila and et. al. [1-3], [25], [27], [34] as a generalization of a semigroup, a generalization of a semihypergroup and a generalization of a Γ-semigroup. A recently book [21] is devoted especially to the study of relationship between hyperstructures and fuzzy sets.

The concept of Γ-rings was introduced by Nobusawa in 1964 [35], as a generalization of the concept of rings. Later Barnes [9] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. After these two papers were published, many mathematicians obtained interesting results on Γ-rings in the sense of Barnes and Nobusawa extending and generalizing many classical notions and results of the theory of rings. Inspired by Nobusawa and Barnes, in 1981 by M.K.Sen in [40] and later in 1986 by Sen and Saha in [41] introduced the notion of Γ-semigroup as a generalization of semigroups and ternary semigroups by taking sets instead of abelian groups. Since then, many classical notions and results of the theory of semigroups have been extended and generalized to Γ-semigroups.

In the framework of hyperstructures, in [29, 30], the concept of $H_v-\Gamma$-semigroup has been introduced and investigated as a generalization of semigroups, Γ-semigroups, semihypergroups, Γ-semihypergroups and H_v-semigroups. Different examples of H_v-Γ-semigroups are presented there.

In this paper, we deal with H_v-Γ-semigroups. Using Atanassov idea, we apply the concept of intuitionistic fuzzy sets to H_v-Γ-semigroups initiating this kind of study. We introduce the notion of an intuitionistic fuzzy H_v-ideal of an H_v-Γ-semigroup and different properties and characterizations of them are investigated and obtained extending some results obtained in H_v-rings. Also some natural equivalence relations on the set of all intuitionistic fuzzy H_v-ideals of an H_v-Γ-semigroup are investigated.

We introduce below necessary notions and present a few auxiliary results that will be used throughout the paper. Recall first the basic terms and definitions from the hyperstructure theory.

A map $\circ: H\times H\to \mathcal{P}^*(H)$ is called *hyperoperation* or *join operation* on the set H, where H is a non-empty set and $\mathcal{P}^*(H)=\mathcal{P}(H)\backslash\{\emptyset\}$ denotes the set of all non-empty subsets of H.

A *hyperstructure* is called the pair $(H,\circ)$ where ? ? ? is a hyperoperation on the set H.

A hyperstructure $(H,\circ)$ is called a *semihypergroup* if $\forall x,y,z\in H$, $(x\circ y)\circ z=x\circ(y\circ z)$, which means that

$$\bigcup_{u\in x\circ y} u\circ z=\bigcup_{v\in y\circ z} x\circ v.$$

If $x\in H$ and A,B are non-empty subsets of H then $A\circ B=\bigcup_{a\in A,b\in B} a\circ b, A\circ x=A\circ\{x\}$, and $x\circ B=\{x\}\circ B$.

A non-empty subset B of a semihypergroup H is called a *sub-semihypergroup* of H if $B\circ B\subseteq B$ and H is called in this case *super-semihypergroup* of B.

Let $(H,\circ)$ be a semihypergroup. Then H is called a *hypergroup* if it satisfies the reproduction axiom, for all $a\in H$, $a\circ H=H\circ a=H$. A non-empty subset I of a semihypergroup H is called a *right* (*left*) *hyperideal* of H if for all $x\in H$ and $r\in I$, $r\circ x\subseteq I(x\circ r\subseteq I)$.

A hypergrupoid $(H,\circ)$ is called an *H_v-group* if for all $x,y,z\in H$ the followig two conditions hold:

1. $(x\circ y)\circ z\cap x\circ(y\circ z)\neq\emptyset$,
2. $x\circ H=H\circ x=H$.

If $(H,\circ)$ satisfies only the first axiom, then it is called an *H_v-semigroup*.

Let X be a non-empty set. A fuzzy subset μ of X is a function $\mu: X\to[0,1]$. Let μ,λ be two fuzzy subsets of X, we say that μ is contained in λ if $\mu(x)\leq\lambda(x),\forall x\in X$.

Atanassov introduced in [5-8] the concept of intuitionistic fuzzy sets defined on a non-empty set X as objects having the form $A=\{\langle x,\mu_A(x),\lambda_A(x)\rangle | x\in X\}$, where the functions $\mu_A: X\to[0,1]$ and $\lambda_A: X\to[0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\lambda_A(x)$) of each element $x\in X$ to the set A respectively, and $0\leq\mu_A(x)+\lambda_A(x)\leq 1$ for all $x\in X$.

Obviously, each ordinary fuzzy set may be written as $A=\{\langle x,\mu_A(x),1-\mu_A(x)\rangle | x\in X\}$.

Let A and B be two intuitionistic fuzzy sets on X. The following expressions are defined in [6, 7].

1. $A\subseteq B$ if and only if $\mu_A(x)\leq\mu_B(x)$ and $\lambda_A(x)\geq\lambda_B(x)$ for all $x\in X$.
2. $A=B$ if and only if if $A\subseteq B$ and $B\subseteq A$.
3. $A^c=\{\langle x,\lambda_A(x),\mu_A(x)\rangle | x\in X\}$.
4. $A\cap B=\{\langle x,\min\{\mu_A(x),\mu_B(x)\},\max\{\lambda_A(x),\lambda_B(x)\}\rangle | x\in X\}$.
5. $A\cup B=\{\langle x,\max\{\mu_A(x),\mu_B(x)\},\min\{\lambda_A(x),\lambda_B(x)\}\rangle | x\in X\}$.
6. $\square A=\{< x,\mu_A(x), 1-\mu_A(x)> | x\in X\}$
7. $\diamond A=\{\langle x,1-\lambda_A(x),\lambda_A(x)\rangle | x\in X\}$.

For the sake of simplicity, we use the symbol $A=(\mu_A,\lambda_A)$ for intuitionistic fuzzy set $A=\{\langle x,\mu_A(x),\lambda_A(x)\rangle | x\in X\}$.

2. SOME NOTIONS IN H_v-Γ-SEMIGROUPS

Recently, in [29,30] $H_v - \Gamma$-semigroups have been introduced and investigated as a generalization of Γ-semigroups, Γ-semihypergroups and H_v-semigroups. We give the definition of H_v-Γ-semigroup in a different way.

Definition 2.1 *Let H and Γ be two non-empty sets. Any map from $H \times \Gamma \times H \to \mathcal{P}^*(H)$ will be called a Γ-hypermultiplication in H and denoted by $(\cdot)_\Gamma$. The result of this hypermultiplication for $a, b \in H$ and $\alpha \in \Gamma$ is denoted by $a\alpha b$. A Γ-semihypergroup H is an ordered pair $(H, (\cdot)_\Gamma)$ where H and Γ are non-empty sets and $(\cdot)_\Gamma$ is a Γ-hypermultiplication on H which satisfies the following property $\forall (a, b, c, \alpha, \beta) \in H^3 \times \Gamma^2, (a\alpha b)\beta c = a\alpha(b\beta c)$.*

Definition 2.2 *Let H and Γ be two non-empty sets. An H_v-Γ-semigroup H is an ordered pair $(H, (\cdot)_\Gamma)$ where H and Γ are non-empty sets and $(\cdot)_\Gamma$ is a Γ-hypermultiplication on H which satisfies the following property*

$$\forall (a, b, c, \alpha, \beta) \in H^3 \times \Gamma^2, ((a\alpha b)\beta c) \cap (a\alpha(b\beta c)) \neq \emptyset.$$

Examples of H_v-Γ-semigroups can be found in [29, 30].

Definition 2.3 *Let H be an H_v-Γ-semigroup. A non-empty subset I of H is called a left (resp. right) H_v-ideal if the following condition holds:* for all $x \in H, y \in I$ and $\alpha \in \Gamma$, $x\alpha y \subseteq I$ (resp., $y\alpha x \subseteq I$).

I is called to be an H_v-ideal of H if it is both a left and a right H_v-ideal of H.

3. INTUITIONISTIC FUZZY H_v-IDEALS

Definition 3.1 *Let H be an H_v-Γ-semigroup. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in H is called a left (resp. right) intuitionistic fuzzy H_v-ideal in H if*

1. $\mu_A(y) \leq \inf\{\mu_A(z) | z \in x\alpha y\}$ (resp., $\mu_A(x) \leq \inf\{\mu_A(z) | z \in x\alpha y\}$, $\forall x, y \in H, \alpha \in \Gamma$,
2. $\sup\{\lambda_A(z) | z \in x\alpha y\} \leq \lambda_A(y)$ (resp. $\sup\{\lambda_A(z) | z \in x\alpha y\} \leq \lambda_A(x)$, $\forall x, y \in H, \alpha \in \Gamma$.

Lemma 3.2 *Let H be an H_v-Γ-semigroup. If $A = (\mu_A, \lambda_A)$ is a left (resp. right) intuitionistic fuzzy H_v-ideal of H, then so is $WA = (\mu_A, \mu_A^c)$.*

Proof. It is sufficient to show that μ_A^c satisfies the condition (2) of Definition 3.1. For $x, y \in H$, we have Let $x, y \in H$ and $\alpha \in \Gamma$. Then since μ_A is a left fuzzy H_v-ideal of H, we have $\mu_A(y) \leq \inf\{\mu_A(z) | z \in x\alpha y\}$, and so $1 - \mu_A^c(y) \leq \inf\{1 - \mu_A^c(z) | z \in x\alpha y\}$, which implies that $\sup\{\mu_A^c(z) | z \in x\alpha y\} \leq \mu_A^c(y)$.

Therefore the condition (2) of Definition 3.1 is verified. The proof of the right H_v-ideals is similar.

Lemma 3.3 *Let H be an H_v-Γ-semigroup. If $A = (\mu_A, \lambda_A)$ is a left (resp. right) intuitionistic fuzzy H_v-ideal of H, then so is $\diamond A = (\lambda_A^c, \lambda_A)$.*

Proof. The proof is similar to the proof of Lemma 3.2, so it is omitted.

Combining the above two lemmas we obtain the following theorem.

Theorem 3.4 *Let H be an H_v-Γ-semigroup. $A = (\mu_A, \lambda_A)$ is a left (resp., right) intuitionistic fuzzy $H - v$-ideal of H if and only if WA and $\diamond A$ are left (resp., right) intuitionistic fuzzy H_v-ideals of H.*

Corollary 3.5 *Let H be an H_v-Γ-semigroup. $A = (\mu_A, \lambda_A)$ is a left (resp., right) intuitionistic fuzzy H_v-ideal of H if and only if μ_A and λ_A^c are left (resp., right) fuzzy H_v-ideals of H.*

For any $t \in [0,1]$ and an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ of an H_v-Γ-semigroup H, the sets

$$U(\mu_A; t) = \{x \in H | \mu_A(x) \geq t\} \text{ and } L(\lambda_A; t) = \{x \in H | \lambda_A(x) \leq t\}.$$

are called respectively an *upper* and *lower t-level* cut of $A = (\mu_A, \lambda_A)$.

Theorem 3.6 *Let H be an H_v-Γ-semigroup. If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy H_v-ideal of H, then for every $t \in Im(\mu_A) \cap Im(\lambda_A)$, the sets $U(\mu_A; t)$ and $L(\lambda_A; t)$ are H_v-ideals of H.*

Proof. Let $t \in Im(\mu_A) \cap Im(\lambda_A) \subseteq [0,1]$. Let $x \in H$, $y \in U(\mu_A; t)$ and $\alpha \in \Gamma$. Since A is a left intuitionistic fuzzy H_v-ideal of H, we have $t \leq \mu_A(y) \leq \inf\{\mu_A(z) | z \in x\alpha y\}$. Therefore, for every $z \in x\alpha y$, we get $\mu_A(z) \geq t$, which implies that $z \in U(\mu_A; t)$, so $x\alpha y \subseteq U(\mu_A; t)$. Now, let $x \in H, y \in L(\lambda_A; t)$ and $\alpha \in \Gamma$. Since A is a left intuitionistic fuzzy H_v-ideal of H, we have $\sup\{\lambda_A(z) | z \in x\alpha y\} \leq \lambda_A(y) \leq t$. Therefore, for all $z \in x\alpha y$, we have $\lambda_A(z) \leq t$, which implies that $z \in L(\lambda_A; t)$, so $x\alpha y \subseteq L(\lambda_A; t)$.

Theorem 3.7 *Let H be an H_v-Γ-semigroup. If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy set of H such that all non-empty levels $U(\mu_A; t)$ and $L(\lambda_A; t)$ are H_v-ideals of H, then $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy H_v-ideal of H.*

Proof. Let $t_1 = \mu_A(y), t_2 = \lambda_A(y)$ for some $x, y \in H$. Then $y \in U(\mu_A; t_1), y \in L(\lambda_A; t_2)$. Since $U(\mu_A; t_1)$ and $L(\lambda_A; t_2)$ are H_v-ideals of H, then for all $\alpha \in \Gamma$, $x\alpha y \subseteq U(\mu_A; t_1)$ and $x\alpha y \subseteq L(\lambda_A; t_2)$. Therefore for every $z \in x\alpha y$, we have $z \in U(\mu_A; t_1)$ and $z \in L(\lambda_A; t_2)$ which imply that $\mu_A(z) \geq t_1$ and $\lambda_A(z) \leq t_2$. Hence, for all $\alpha \in \Gamma$,

$$\inf\{\mu_A(z) | z \in x\alpha y\} \geq t_1 = \mu_A(y) \text{ and } \sup\{\lambda_A(z) | z \in x\alpha y\} \leq t_2 = \lambda_A(y).$$

This completes the proof of theorem.

Corollary 3.8 *Let H be an H_v-Γ-semigroup and I be a left H_v-ideal of H. If fuzzy sets μ and λ are defined on H by* $\mu(x) = \begin{cases} p_0 & if\ x \in I, \\ p_1 & if\ x \in H \setminus I \end{cases}$ *and* $\lambda(x) = \begin{cases} q_0 & if\ x \in I, \\ q_1 & if\ x \in H \setminus I \end{cases}$. *where* $0 \le p_1 < p_0, 0 \le q_0 < q_1$, *and* $p_i + q_i \le 1$ *for* $i = 0,1$, *then* $A = (\mu, \lambda)$ *is an intuitionistic fuzzy* H_v*-ideal of* H *and* $U(\mu; p_0) = I = U(\lambda; q_0)$.

Corollary 3.9 *Let H be an H_v-Γ-semigroup and I be a left H_v-ideal of H. If χ_I is the characteristic function of a I, then $A = (\chi_I, \chi_I^c)$ is a left intuitionistic fuzzy H_v-ideal of H.*

Theorem 3.10 *Let H be an H_v-Γ-semigroup. If $A = (\mu_A, \lambda_A)$ is a left intuitionistic fuzzy H_v-ideal of H, then for all* $x \in H$, $\mu_A(x) = \sup\{t \in [0,1] | x \in U(\mu_A; t)\}$ and $\lambda_A(x) = \inf\{t \in [0,1] | x \in L(\lambda_A; t)\}$.

Proof. Let $w = \sup\{t \in [0,1] | x \in U(\mu_A; t)\}$ and let be an arbitrary $s > 0$. Then $w - s < t$ for some $t \in [0,1]$ such that $x \in U(\mu_A; t)$. This means that $w - s < \mu_A(x)$ so that $w \le \mu_A(x)$ since s is arbitrary.
Now, if $\mu_A(x) = c$, then $x \in U(\mu_A; c)$, and so $c \in \{t \in [0,1] | x \in U(\mu_A; t)\}$.
Hence $\mu_A(x) = c \le \sup\{t \in [0,1] | x \in U(\mu_A; t)\} = w$. Therefore $\mu_A(x) = w = \sup\{t \in [0,1] | x \in U(\mu_A; t)\}$.
Now, let $q = \inf\{t \in [0,1] | x \in L(\lambda_A; t)\}$. Then $\inf\{t \in [0,1] | x \in L(\lambda_A; t)\} < q + s$, for any $s > 0$, and so $t < q + s$ for some $t \in [0,1]$ with $x \in L(\lambda_A; t)$. Since $\lambda_A(x) \le t$ and s is arbitrary, it follows that $\lambda_A(x) \le q$.
Let now $\lambda_A(x) = p$. Then $x \in L(\lambda_A; p)$, and thus $p \in \{t \in [0,1] | x \in L(\lambda_A; t)\}$. Hence $\inf\{t \in [0,1] | x \in L(\lambda_A; t)\} \le p$, that is, $q \le p = \lambda_A(x)$. Consequently $\lambda_A(x) = q = \inf\{t \in [0,1] | x \in L(\lambda_A; t)\}$, which completes the proof.

Definition 3.11 *Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be intuitionistic fuzzy H_v-sub-Γ-semigroups of an H_v-Γ-semigroup H. Then, $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy H_v-ideal of $B = (\mu_B, \lambda_B)$ if $A \subseteq B$ and*

(i) $\min\{\mu_A(x), \mu_B(y)\} \le \inf\{\mu_A(x) | z \in x\gamma y\}$ and $\max\{\lambda_A(x), \lambda_B(y)\} \ge \sup\{\lambda_A(x) | z \in x\gamma y\}$,

(ii) $\min\{\mu_B(x), \mu_A(y)\} \le \inf\{\mu_A(x) | z \in x\gamma y\}$ and $\max\{\lambda_B(x), \lambda_A(y)\} \ge \sup\{\lambda_A(x) | z \in x\gamma y\}$,

for all $x, y \in H$ and $\gamma \in \Gamma$. If $B = (\mu_B, \lambda_B) = (\mathcal{X}_H, \mathcal{X}_H^c)$, then $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy H_v-ideal of H.

Theorem 3.12 *Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be intuitionistic fuzzy H_v-sub-Γ-semigroups of an H_v- Γ-semigroup H such that $A \subseteq B$. Then, $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy H_v-ideal of $B = (\mu_B, \lambda_B)$ if and only if for any $t, s \in (0,1]$, if $A_{(t,s)} \neq \emptyset$, if $A_{(t,s)}$ is a hyperideal of $B_{(t,s)}$.*

Proof. Assume that $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy H_v-ideal of $B = (\mu_B, \lambda_B)$. Then we prove that $A_{(t,s)} \Gamma B_{(t,s)} \subseteq A_{(t,s)}$ and $B_{(t,s)} \Gamma A_{(t,s)} \subseteq A_{(t,s)}$. Let $x \in A_{(t,s)}$ and $y \in B_{(t,s)}$, since $z \in x\gamma y \subseteq A_{(t,s)} \Gamma B_{(t,s)}$ for $\gamma \in \Gamma$, so we have $\mu_A(x) \ge t$ and $\lambda_A(x) \le s$, and $\mu_B(y) \ge t$ and $\lambda_B(y) \le s$ this implies $\min\{\mu_A(x), \mu_B(y)\} \ge t$ and $\max\{\lambda_A(x), \lambda_B(y)\} \le s$. Thus, $\mu_A(z) \ge t$ and $\lambda_A(z) \le s$ for $z \in x\gamma y$ this implies $z \in A_{(t,s)}$. Thus, $A_{(t,s)} \Gamma B_{(t,s)} \subseteq A_{(t,s)}$. Similarly, we can prove $B_{(t,s)} \Gamma A_{(t,s)} \subseteq A_{(t,s)}$.
Conversely, suppose that $A_{(t,s)}$ is a hyperideal of $B_{(t,s)}$. Let for all $x, y \in H$, we put $t_0 = \min\{\mu_A(x), \mu_B(y)\}$ and $s_0 = \max\{\lambda_A(x), \lambda_B(y)\}$. Then, $\mu_A(x) \ge t_0$, $\mu_B(y) \ge t_0$ and $\lambda_A(x) \le s_0$, $\lambda_B(y) \le s_0$ this implies that $x \in A_{(t_0,s_0)}$ and $y \in B_{(t_0,s_0)}$. Since $A_{(t,s)} \Gamma B_{(t,s)} \subseteq A_{(t,s)}$, so we have $x\gamma y \subseteq A_{(t_0,s_0)}$. Thus, for all $z \in x\gamma y$, we get $\mu_A(z) \ge t_0$ and $\lambda_A(z) \le s_0$. This implies $\mu_A(z) \ge t_0 = \min\{\mu_A(x), \mu_B(y)\}$ and $\lambda_A(z) \le s_0 = \max\{\lambda_A(x), \lambda_B(y)\}$. Therefore, this proves the first condition of Definition 3.11. Similarly, we can prove the second condition of Definition 3.11.

Definition 3.13 *Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be intuitionistic fuzzy sets of an H_v-Γ-semigroup H. Then, the product of A and B, denote as $A * B = (\mu_{A*B}, \lambda_{A*B})$, is defined by:*

$$\mu_{A*B}(r) = \begin{cases} \bigvee_{r \in p\gamma q} \{\mu_A(p) \wedge \mu_B(q)\} & \text{if } r \in p\gamma q \\ 0 & \text{Otherwise} \end{cases} \quad \text{and} \quad \lambda_{A*B}(r) = \begin{cases} \bigwedge_{r \in p\gamma q} \{\lambda_A(p) \vee \lambda_B(q)\} & \text{if } r \in p\gamma q \\ 1 & \text{Otherwise} \end{cases}$$

for all $p, q \in H$ and $\gamma \in \Gamma$.

Theorem 3.14 *Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be intuitionistic fuzzy H_v-sub-Γ-semigroups of an H_v-Γ-semigroup H. Then A is an intuitionistic fuzzy H_v-ideal of B if and only if $A * B \subseteq A$ and $B * A \subseteq A$.*

Proof. The proof of this theorem is easy, we omit.

Definition 3.15 *An H_v-Γ-semigroup is said to have intuittionistic fuzzy H_v-ideal extension property if for each intuitionistic fuzzy H_v-sub-Γ-semigroups $B = (\mu_B, \lambda_B)$ and each intuitionistic fuzzy H_v-ideal $A = (\mu_A, \lambda_A)$ of $B = (\mu_B, \lambda_B)$, there exists an intuitionistic fuzzy H_v-ideal $C = (\mu_C, \lambda_C)$ of H such that $C \cap B = A$.*

Particularly, for each H_v-sub-Γ-semigroups T of H and for each intuitionistic fuzzy H_v-ideal $B = (\mu_B, \lambda_B)$ of T, there exists an intuitionistic fuzzy H_v-ideal $C = (\mu_C, \lambda_C)$ of H such that $C \cap \mathcal{X} = B$, where $\mathcal{X} = (\mathcal{X}_T, \mathcal{X}_T^c)$, then H is called to have the strongly intuitionistic fuzzy H_v-ideal extension property.

Theorem 3.16 *An H_v-Γ-semigroup H has the intuitionistic fuzzy H_v-ideal extenstion property if each H_v-sub-Γ-semigroups T of H has the intuitionistic fuzzy H_v-ideal extension property.*

Proof. Suppose that T is a H_v-sub-Γ-semigroups of H this implies T is an H_v-Γ-semigroup. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy H_v-sub-Γ-semigroups of T and $B = (\mu_B, \lambda_B)$ be an intuitionistic fuzzy H_v-ideal of $A = (\mu_A, \lambda_A)$. Define $A^* = (\mu_{A^*}, \lambda_{A^*})$ and $B^* = (\mu_{B^*}, \lambda_{B^*})$ as following

$$\mu_{A^*}(x)=\begin{cases}\mu_A(x) & \text{if } x\in T\\ 0 & \text{if } x\notin T\end{cases} \text{ and } \lambda_{A^*}(x)=\begin{cases}\lambda_A(x) & \text{if } x\in T\\ 0 & \text{if } x\notin T\end{cases}$$

and

$$\mu_{B^*}(x)=\begin{cases}\mu_B(x) & \text{if } x\in T\\ 0 & \text{if } x\notin T\end{cases} \text{ and } \lambda_{B^*}(x)=\begin{cases}\lambda_B(x) & \text{if } x\in T\\ 0 & \text{if } x\notin T\end{cases}$$

Thus, $A^*=(\mu_{A^*},\lambda_{A^*})$ is an intuitionistic fuzzy H_v-sub-Γ-semigroups of H and $B^*=(\mu_{B^*},\lambda_{B^*})$ is an intuitionistic fuzzy H_v-ideal of $A^*=(\mu_{A^*},\lambda_{A^*})$. Thus, by hypothesis, there exists an intuitionistic fuzzy H_v-ideal $C=(\mu_C,\lambda_C)$ of H such that $C\cap A^*=B^*$. This implies $C|_T\cap A=B$, where $C|_T$ is a ristriction of C in T. Clearly, this is an intuitionistic fuzzy H_v-ideal of H as H_v-Γ-semigroup. Therefore, we achieve that T has the intuitionistic fuzzy H_v-ideal extension property.

The converse part is obvious.

Theorem 3.17 *An H_v-Γ-semigroup H has the intuitionistic fuzzy H_v-ideal extenstion property if each hmomorphic image of H has the intuitionistic fuzzy H_v-ideal extension property.*

Proof. Assume that Π is a surjective homomorphism from H on to H^*. Let $A^*=(\mu_{A^*},\lambda_{A^*})$ is an intuitionistic fuzzy H_v-sub-Γ-semigroupsp of H^* and $B^*=(\mu_{B^*},\lambda_{B^*})$ is an intuitionistic fuzzy H_v-ideal of $A^*=(\mu_{A^*},\lambda_{A^*})$. Let $A=\Pi^{-1}(A^*)$ and $B=\Pi^{-1}(B^*)$. Then, clearly $A=(\mu_A,\lambda_A)$ is an intuitionistic fuzzy H_v-sub-Γ-semigroups of H and $B=(\mu_B,\lambda_B)$ is an intuitionistic fuzzy H_v-ideal of $A=(\mu_A,\lambda_A)$. Since H has the intuitionistic fuzzy H_v-ideal extension property, then there exists an intuitionistic fuzzy H_v-ideal $C=(\mu_C,\lambda_C)$ of H such that $C\cap B=A$. Let $C^*=\Pi(C)$. Then, C^* is an intuitionistic fuzzy H_v-ideal of H^*. We have

$$\Pi^{-1}(C^*\cap B^*)=\Pi^{-1}(C^*)\cap\Pi^{-1}(B^*)=C\cap B=A=\Pi^{-1}(A^*).$$

Hence, we have $C^*\cap B^*=\Pi\Pi^{-1}(C^*\cap B^*)=\Pi\Pi^{-1}(A^*)=A^*$ this implies $C^*\cap B^*=A^*$.

4. THE RELATIONS $\mathfrak{U}^a$ AND $\mathfrak{L}^a$

Let $a\in[0,1]$ be fix and let $IF(H)$ be the family of all intuitionistic fuzzy left H_v-ideals of an H_v-Γ-semigroup H. For any $A=(\mu_A,\lambda_A)$ and $B=(\mu_B,\lambda_A)$ from $IF(H)$, we define two binary relations $\mathfrak{U}^a$ and $\mathfrak{L}^a$ on $IF(H)$ as follows: $(A,B)\in\mathfrak{U}^a\Leftrightarrow U(\mu_A;a)=U(\mu_B;a)$, $(A,B)\in\mathfrak{L}^a\Leftrightarrow L(\lambda_A;a)=L(\lambda_B;a)$. The two relations $\mathfrak{U}^a$ and $\mathfrak{L}^a$ are equivalent relations. Hence $IF(H)$ can be divided into equivalence classes of $\mathfrak{U}^a$ and $\mathfrak{L}^a$, denoted by $[A]_{\mathfrak{U}^a}$ and $[A]_{\mathfrak{L}^a}$ for any $A=(\mu_A,\lambda_A)\in IF(H)$, respectively. The corresponding quotient sets will be denoted as $IF(H)/\mathfrak{U}^a$ and $IF(H)/\mathfrak{L}^a$, respectively. For the family $LI(H)$ of all left H_v-ideals of H, we define two maps U_a and L_a from $IF(H)$ to $LI(H)\cup\{\emptyset\}$ putting $U_a(A)=U(\mu_A;t), L_a(A)=L(\lambda_A;t)$, for each $A=(\mu_A,\lambda_A)\in IF(H)$.

It can be easily seen that these maps are well-defined.

Lemma 4.1 *For any $a\in(0,1)$, the maps U_a and L_a are surjective.*

Proof. Let 0 and 1 be fuzzy sets on H defined by $0(x)=0$ and $1(x)=1$ for all $x\in H$. Then $0_{:}=(0,1)\in IF(H)$ and $U_a(0_{:})=L_a(0_{:})=\emptyset$ for any $a\in(0,1)$. Moreover, for any $K\in LI(H)$, we have $I_{:}=(\chi_K,\chi_K^c)\in IF(H), U_a(I_{:})=U(\chi_K;a)=K$, and $L_a(I_{:})=L(\chi_K^c;a)=K$. Hence U_a and L_a are surjective.

Theorem 4.2 *For any $a\in(0,1)$, the sets $IF(H)/\mathfrak{U}^a$ and $IF(H)/\mathfrak{L}^a$ are equipotent to $LI(H)\cup\{\emptyset\}$.*

Proof. Let $a\in(0,1)$. Putting $U_a^*([A]_{\mathfrak{U}^a})=U_a(A)$ and $L_a^*([A]_{\mathfrak{L}^a})=L_a(A)$ for any $A=(\mu_A,\lambda_A)\in IF(H)$, we obtain two maps: $U_a^*: IF(H)/\mathfrak{U}^a\to LI(H)\cup\{\emptyset\}$, $L_a^*: IF(H)/\mathfrak{L}^a\to LI(H)\cup\{\emptyset\}$.

If $U(\mu_A;a)=U(\mu_B;a)$ and $L(\lambda_A;a)=L(\lambda_B;a)$ for some $A=(\mu_A,\lambda_A)$ and $B=(\mu_B,\lambda_B)$ from $IF(H)$, then $(A,B)\in\mathfrak{U}^a$ and $(A,B)\in\mathfrak{L}^a$, whence $[A]_{\mathfrak{U}^a}=[B]_{\mathfrak{U}^a}$ and $[A]_{\mathfrak{L}^a}=[B]_{\mathfrak{L}^a}$, which means that U_a^* and L_a^* are injective.

We will show that the maps U_a^* and L_a^* are surjective. Let $K\in LI(H)$. Then for $I_{:}=(\chi_K,\chi_K^c)\in IF(H)$, we have $U_a^*([I_{:}]_{\mathfrak{U}^a})=U(\chi_K;a)=K$ and $L_a^*([I_{:}]_{\mathfrak{L}^a})=L(\chi_K^c;a)=K$. Also $0_{:}=(0,1)\in IF(H)$. Moreover, $U_a^*([0_{:}]_{\mathfrak{U}^a})=U(0;a)=\emptyset$ and $L_a^*([0_{:}]_{\mathfrak{L}^a})=L(1;a)=\emptyset$. Hence U_a^* and L_a^* are surjective.

Now for any $a\in[0,1]$, we have the new relation $\mathfrak{H}^a$ on $IF(H)$ putting

$$(A,B)\in\mathfrak{H}^a\Leftrightarrow U(\mu_A;a)\cap L(\lambda_A;a)=U(\mu_B;a)\cap L(\lambda_B;a),$$

where $A=(\mu_A,\lambda_A)$ and $B=(\mu_B,\lambda_B)$. It can be seen easily that $\mathfrak{H}^a$ is an equivalence relation.

Lemma 4.3 *The map $I_a: IF(H)\to LI(H)\cup\{\emptyset\}$ defined by $I_a(A)=U(\mu_A;a)\cap L(\lambda_A;a)$, where $A=(\mu_A,\lambda_A)$ is surjective for any $a\in(0,1)$.*

Proof. Let $a\in(0,1)$ be fixed. Then for $0_{:}=(0,1)\in IF(H)$, we have $I_a(0_{:})=U(0;a)\cap L(1;a)=\emptyset$, and for any $K\in LI(H)$, there exists $I_{:}=(\chi_K,\chi_K^c)\in IF(H)$ such that $I_a(I_{:})=U(\chi_K;a)\cap L(\chi_K^c;a)=K$.

Theorem 4.4 *For any $a\in(0,1)$, the quotient set $IF(H)/\mathfrak{U}^a$ is equipotent to $LI(H)\cup\{\emptyset\}$.*

Proof. Let $I_a^*: IF(H)/\mathfrak{U}^a\to LI(H)\cup\{\emptyset\}$, where $a\in(0,1)$, be defined by the formula

$$I_a^*([A]_{\mathfrak{H}^a})=I_a(A) \text{ for every } [A]_{\mathfrak{H}^a}\in IF(H)/\mathfrak{H}^a.$$

If $I_a^*([A]_{\mathfrak{H}^a}) = I_a^*([B]_{\mathfrak{H}^a})$ for some $[A]_{\mathfrak{H}^a}, [B]_{\mathfrak{H}^a} \in IF(H)/\mathfrak{H}^a$, then $U(\mu_A; a) \cap L(\lambda_A; a) = U(\mu_B; a) \cap L(\lambda_B; a)$, which implies that $(A, B) \in \mathfrak{H}^a$ and as consequence, $[A]_{\mathfrak{H}^a} = [B]_{\mathfrak{H}^a}$. Thus I_a^* is injective. It is also onto because $I_a^*(0_:) = I_a(0_:) = \emptyset$ for $0_: = (0,1) \in IF(H)$, and $I_a^*(I_:) = I_a(K) = K$ for $K \in LI(H)$ and $I_: = (\chi_K, \chi_K^c) \in IF(H)$.
The main tools in the theory of H_v-structures are the fundamental relations. Similar to [30] with the necessary adoptions, the relation ξ^* is the smallest equivalence relation on H such that the quotient H/ξ^* is a Γ/ξ^*-semigroup. ξ^* is called the fundamental equivalence relation on H. In similar way, according [30], if $\mathcal{U}$ denotes the set of all expressions consisting of finite -hyperoperation on H (that is, $\mathcal{U} = \{a_1\gamma_1 a_2\gamma_2 \cdots a_n\gamma_n a_{n+1} | a_i \in S, \gamma_i \in \Gamma, \forall i \in \{1, \dots, n\}, n \in N\}$), then a relation ξ can be defined on H as follows: $x\xi y \Leftrightarrow \{x, y\} \subseteq u$ for some $u \in \mathcal{U}$.
In similar way, according [30] with the necessary adoptions, the transitive closure of ξ is the fundamental relation ξ^*, that is, $a\xi^* b$ if and only if there exist $x_1, \dots, x_{m+1} \in H; u_1, \dots, u_m \in \mathcal{U}$ with $x_1 = a, x_{m+1} = b$ such that $\{x_i, x_{i+1}\} \subseteq u_i$, $(i = 1, \dots, m)$. Let us suppose that $\xi^*(a)$ is the equivalence class containing $a \in H$. Then the product e on H/ξ^* is defined as follows: $\xi^*(a) \mathrm{e} \xi^*(b) = \xi^*(d)$, $\forall d \in \xi^*(a)\gamma/\xi^*\xi^*(b)$, for all $\gamma/\xi^* \in \Gamma/\xi^* = \{\gamma/\xi^* | \gamma \in \Gamma\}$ where $\xi^*(a)\gamma/\xi^*\xi^*(b) = \{\xi^*(c) | c \in a\gamma b\}$. Then it is well-known that $(H/\xi^*, \mathrm{e})$ is a Γ/ξ^*-semigroup.
Definition 4.5 *Let H be an H_v-Γ-semigroup and μ a fuzzy subset of H. The fuzzy subset μ_{ξ^*} on H/ξ^* is defined as follows:* $\mu_{\xi^*}: H/\xi^* \to [0,1]$, $\mu_{\xi^*}(\xi^*(x)) = \sup\{\mu(a) | a \in \xi^*(x)\}$.
Theorem 4.6 *Let H be an H_v-Γ-semigroup and $A = (\mu_A, \lambda_A)$ a left intuitionistic fuzzy H_v-ideal of H. Then $A/\xi^* = (\mu_{\xi^*}, \lambda_{\xi^*})$ is a left intuitionistic fuzzy ideal of H/ξ^*.*

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