DYNAMICAL TEACHING FOR THE NEW AGE: EXAMPLE OF THE TEACHING AND LEARNING OF FEATURES OF DERIVATIVE

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Abstract: Differential calculus represents a very important field of mathematics, primarily because of the possibility of its application. The derivative of a function, a key element of differential calculus, finds its application not only in mathematics but also in most other natural sciences. By its very nature, differential calculus is an abstract and, to understand, rather complicated and demanding theory for students. Due to the presence of highly abstract concepts, students have difficulties understanding the true meaning of differential calculus, and especially seeing a little further, i.e. recognizing situations where that theory can be applied. In higher education, differential calculus, primarily the derivative of a function, is an unavoidable moment in the teaching of mathematics, and it is very important how it will be presented to students. In teaching mathematics, it is very important that students have properly formed ideas about mathematical concepts. This is especially related to the concept of the derivative of a function, which is necessary for students not only as a concept that occurs in the teaching of mathematics but as a necessary tool for many other fields and sciences. Successful organization of the process of formation of mathematical concepts and successful management of their adoption can be achieved through the modernization of the teaching process. Special attention in this paper is devoted to the improvement of the teaching process through the application of modern computer technologies. The *GeoGebra* software, which was used for the purpose of preparing the material for this work, is described and shown. The *GeoGebra* software was chosen for its ability to create animations and simulations. The dynamic nature of the *GeoGebra* software answer the demands of modern teaching and fits perfectly into the students' habits for the active learning process. For the purposes of this work, a dynamical material containing moving images, illustrations, and graphs which through its interesting story illustrates the features of the derivative of the function in a very interesting way was designed by the authors of the paper. Presenting the basic content related to the concept of the derivative of a function in this way enables students to understand and acquire these concepts more quickly. The material presented in the paper was realized using *GeoGebra* software. All the elements of the used *GeoGebra* material were explained in detail. The impressions of teachers and students about the effects of applying such materials in teaching mathematics and science were more than positive. Our mission for the future is to continue with the development and improvement of teaching materials and techniques by using modern technologies.
Keywords: Differential calculus, derivative, *GeoGebra*, dynamical materials.

1. INTRODUCTION

Modern computer technologies have a significant impact on mathematics education, its content, methods, and techniques, curricula, and programs, as well as on the entire organization of teaching and learning processes. In modern mathematics education, the question is no longer whether computer technologies should be applied, but the main task is to find optimal solutions on how and in what way to apply new technologies in teaching practice, in order to improve the quality of teaching and make learning more efficient.

The use of computers in the learning process can be of significant help to students when calculating, drawing graphs, working with larger data sets, navigating the world of symbols, connecting different representations of mathematical objects, experimenting, making assumptions, and checking their correctness. Computer technologies allow students to spend more time thinking about mathematical ideas, researching, and discovering mathematical knowledge. The potential of computer technologies to connect mathematical concepts with visual representations in a way that encourages mathematical reasoning and conceptual understanding is very important for mathematics teaching/learning (Lopez Jr, 2001). For modern educational technology, there are almost no obstacles to visually displaying any mathematical or real object and adapting them according to the sensory and cognitive capabilities of students: "*While real-life objects become abstract on screen, mathematical objects that are abstract on screen become concrete*" (Lester, 2000).

In this paper, we will demonstrate the application of computer technologies in teaching and learning mathematics. On the example of teaching Derivative of a function and its features, we will develop the dynamical teaching materials realized in *GeoGebra* software. The teaching and learning process and the use of *GeoGebra* software, and also the created dynamical materials are described in detail, as also the reactions of students and the impressions of teachers.

2. MATERIALS AND METHODS

Dynamic visualization provides an opportunity to show students some aspects of mathematics that are extremely difficult, if not impossible, to demonstrate using pen and paper. It is difficult to imagine how dynamic processes would be depicted without computers and how long it would take us to achieve the same goal. Taking into account that the derivative of a function is one of the most abstract fields of mathematics for students, and on the other hand, is one of the most applicable fields of mathematics, we came to the idea to use the possibilities of dynamic visualization and computer technologies in order to improve the teaching process and achievements of the students.

GeoGebra software

GeoGebra software is designed specifically for educational purposes, is intuitive and easy to use, and provides a wide range of possibilities for the effective implementation of modern methodological approaches. *GeoGebra* is a dynamic mathematical software that combines the ease of use and constructive features of DGS (Dynamical Geometrical Systems) with the power and functionality of CAS (Computer Algebra Systems), thus bridging the gap between the mathematical disciplines of geometry, algebra, and mathematical analysis (Hohenwarter & Preiner, 2007). Because of its versatility, *GeoGebra* can be used at all levels of education for a wide range of different mathematical topics (Hohenwarter, Hohenwarter & Lavicza, 2008).

GeoGebra has several interconnected views: graphical, algebraic, tabular, and a window for symbolic calculations. Mathematical objects are created using appropriate tools and are represented by multiple representations: graphical (e.g. graphs of functions, points, lines, and other geometric objects), algebraic (e.g. coordinates of points, equations), and numerical representations (in table cells). In doing so, the user can directly influence the change of any representation. All representations are dynamically linked and automatically change when any of them change, regardless of how the object was originally created. In addition, static graphic representations or geometric constructions can be animated into a moving image - a dynamic visual representation, which reveals new relationships between mathematical objects that are much more difficult (or impossible) to represent using static images. Interactive manipulation of objects and dynamic connection of multiple representations provide the possibility of displaying mathematical objects, geometric and algebraic transformations, as well as dynamic mathematical processes in a visual environment. *GeoGebra* is a powerful tool for dynamic visualization and works with multiple representations and allows the teacher to effectively adapt teaching content and teaching methods in order to actualize students' visual thinking and thereby facilitate the connection of visual knowledge with the formal-symbolic language of mathematics (Verhoef, Coenders, Pieters, van Smaalen & Tall, 2015).

GeoGebra is open-source software, available to all users on its official website www.geogebra.org. It is very interesting, that the *GeoGebra* website not only offers applications to download but also gathers the *GeoGebra* community and allows sharing and using already prepared materials which is very useful for both, students and teachers.

The application of GeoGebra materials is now more than common, especially concerning the new trends of educational practice imposed by online teaching and learning in the pandemic conditions (Kostić & Sekulić, 2022; Sekulić & Kostić, 2022; Sekulić, Manigoda & Kostić, 2022).

Teaching and learning the derivative of a function

The main problem in the graphical understanding of the derivative is the relationship between the slope of the tangent to the graph of the function and the derivative of the function at a point. This is the basis for understanding the derivative function, which, among other things, gives for each point in the domain the corresponding value of the slope of the tangent.

For a graphical understanding of the derivative of a function, students need to have appropriate knowledge of the following graphical representations: 1. graph of the function, 2. tangent to the graph of the function, 3. graphical representation of the slope of the tangent and 4. graph of the derived function. In addition, switching between two graphical representations also requires knowledge that establishes reference links between those representations.

In order to overcome conceptual obstacles in processing the concepts of mathematical analysis, much attention is paid to modern research on visual approaches supported by computer technologies (Sekulic *et al*. 2020; Sekulić, Borjanović & Popović, 2022). One of the well-known approaches to introducing concepts of calculus/mathematical analysis was given by David Tall (2010) and called the "sensible approach" to calculus. The main advantage of the sensible approach is that concepts that are known to cause difficulties for students (e.g. boundary processes) are not approached formally in the initial stages of learning, but through perceptual and intuitive knowledge the fundamental ideas of calculus are naturally developed in a way to build cognitive the basis for their symbolization and then formalization in the further course of the learning process. For the adoption of a formal definition of the derivative of a function, the cognitive basis is represented by the dynamic visual idea of "local straightness", which is based on the nature of human visual perception and is reflected in the following: enlarging the part of the graph of the function located in the vicinity of one of its points, one can see how the graph looks less and less "curved" and under high magnification, it looks like a straight line. This allows the student to "see" the slope of that straight line and thus build a geometric sense of the concept of the slope of a curve, and then the concept of the tangent to a curve.

Method

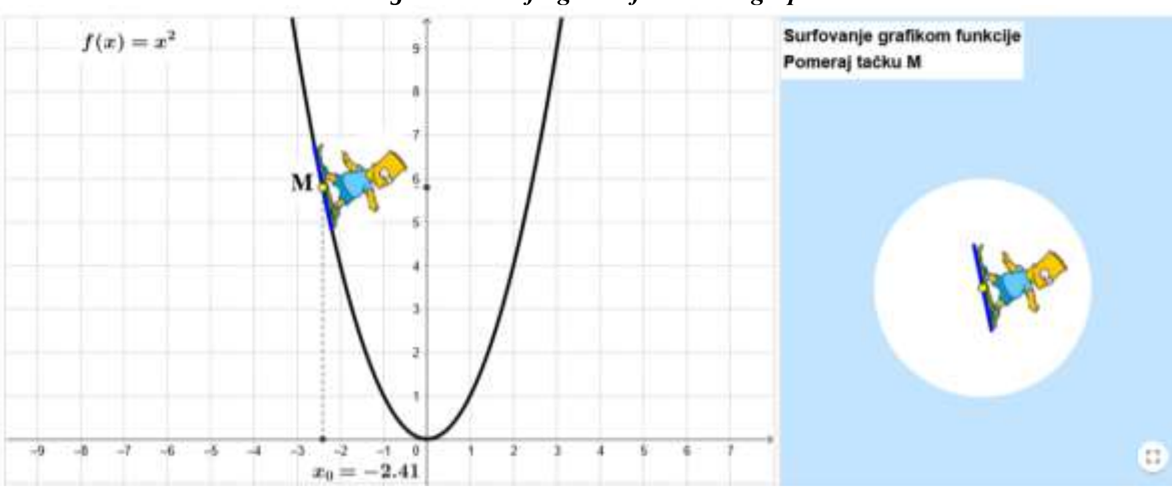
For teaching the concept of the first derivative of a function, the authors of the paper propose a cognitive-visual concept implemented in the *GeoGebra* environment and based on the visualization of the learning process in accordance with the basic ideas of a sensitive approach to calculus. The cognitive-visual approach is designed so that in the processing of the derivative of a function, the concept of the derivative of a function at a point is first introduced, then the concept of the tangent of a graph.

In the following text, the introduction of the definition of the first derivative of a function at a point will be described, using a cognitive-visual approach. The starting point in processing the concept of the derivative of a function is the concept of the slope of a curve, which is introduced and based on previous knowledge and intuitive ideas that students have about it. In this initial phase of learning, students need to gain insight into the concept of the slope of a curve at an intuitive, enactive and visual level of abstraction.

For the implementation of teaching activities, pre-prepared *GeoGebra* teaching materials are used, with which students work independently. All materials are developed by the authors of the paper and are available for all users on the official *GeoGebra* website. For the purpose of teaching the derivative of a function, two *GeoGebra* materials were initially made.

The first material created is a *GeoGebra* simulation of movement along the graph of a function, (Figure 1). In this material, a popular character from a cartoon is intentionally included, to make students interested in manipulating him and seeing how mathematical content can be recognized in a real-life environment. Students "surf" the function's graph by manipulating an object (the cartoon character) on the screen with the mouse, by dragging it along the curve.

Figure 1. "Surfing" the function's graph

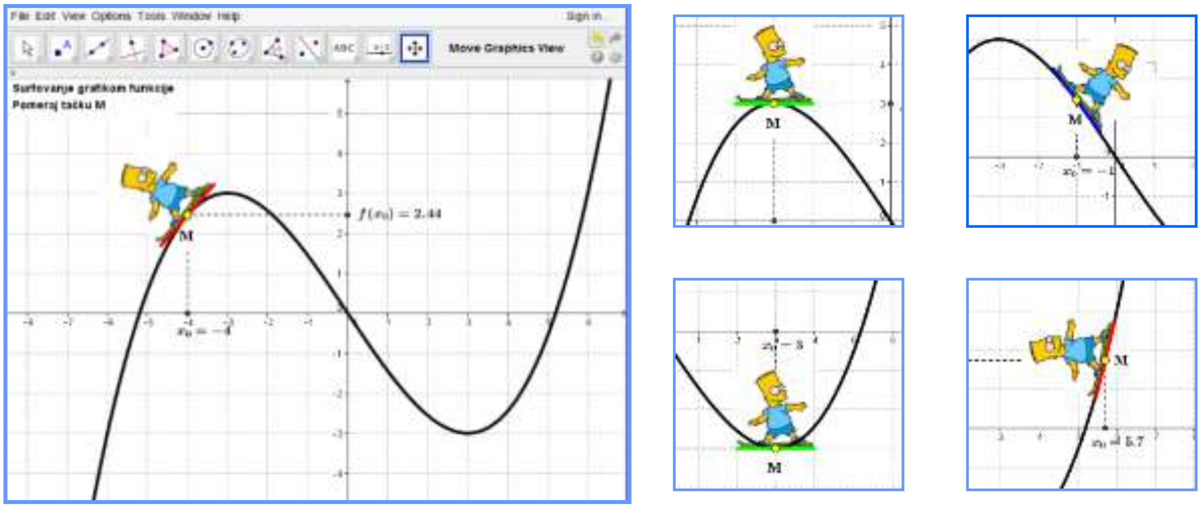


Students see the graph as a physical object and when they move a point along the graph, they can feel the change in the slope of the curve through sensorimotor activities. Surfing in the *GeoGebra* environment supports gaining insight into the change in the slope of a curve, with the end result that students can make inferences about some qualitative properties of the slope (e.g. the slope is positive/negative, the slope increases/decreases), simply by "looking along the curve", without numerical calculation or symbolic manipulation.

The material has two windows, the left for the representation of the function's graph, the cartoon character on the skate, and the point M which is used for moving the character along the function. The function can be changed so that the students can observe different functions and the slopes of tangents in their points.

In the right window is represented only the cartoon character on the skate, in order for students to see more precisely its movement direction: up, down or horizontal. Each separate movement direction is marked in different colors: red (for going up), blue (for going down), and green (for horizontal position, equally, the position of the function's extreme value), (Figure 2).

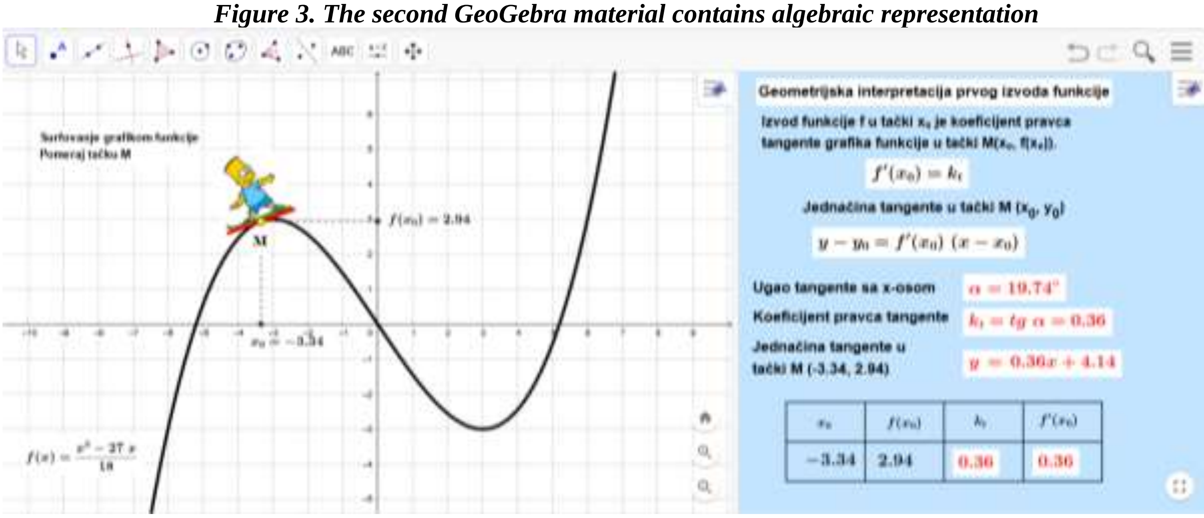
Figure 2. Simulation of movement along the graph of a function



The presented *GeoGebra* material is available at the official *GeoGebra* website: (https://www.geogebra.org/m/uhz4vfek).

The concept of the slope of a curve is further elaborated on a more subtle level, by relating it to the algebraic representation of the slope and tangent of a curve. For this purpose, the second *GeoGebra* teaching material was prepared. In this material, the cartoon character remains present, with the addition of the algebraic representation (Figure 3).

Figure 3. The second GeoGebra material contains algebraic representation



It can be seen that this material also has two windows. The left window remains the same as in the previous material, but the right window is now updated with the algebraic representation and mathematical formulas of the first derivative relationship with the slope of the tangent line, and the equation of the tangent line.

Also, the right window is dynamically connected with the movements of the cartoon character in the left window and the displayed values of the angle between the tangent line and x-Axis, the value of the slope of the tangent line, and the corresponding equation of the tangent line. The table at the bottom of the right window displays the values of the coordinates of point M from the function, the value of the slope of the tangent line, and the value of the first derivative at point M. The last two values are marked in red color so that the students could be able to perceive the

connection between the first derivative and the slope of the tangent line at the same point. Those values change, as the position of the cartoon character is changed in the left window.
This material is also available on the official *GeoGebra* website, for download or for immediate use (https://www.geogebra.org/m/ffcqvbk6).
Both materials could also be used for teaching the monotonicity of function and extreme values because it allows students to explore different positions and by the movement of the cartoon character could make conclusions about the monotonicity of the function.

3. RESULTS AND DISCUSSION

In the work with the students, a cognitive-visual approach was applied to the study of the derivative of a function. The main impression about the application of presented dynamical *GeoGebra* materials was that the students were more than happy to use them during the teaching and learning process. Observing the students during the class it was obvious that they could work independently with teaching materials in a computer environment. Also, the students were satisfied due to active participation in joint teaching activities, stimulating work atmosphere, interest, and motivation during classes. Learning about the derivative of a function in a computer environment using innovative teaching materials with visual representations and the chosen approach used the potential of students' visual thinking and enabled the development of their abilities to construct and interpret different representations, thereby to build the necessary structures for a complete and essential understanding of the derivative of a function in different ways.
The teachers' notions were that the cognitive-visual approach to learning the derivative of a function in the computer environment was more than successful because the visual and dynamic representations brought to students closer the understanding of the concept of the derivative of a function. Also, in discussion with the other mathematics teachers, they showed quite impressive interest in the presented teaching materials, in order to use them in their teaching practice. The exchange of teaching materials was regular among mathematics teachers after our experience, which led to improving our practice and the learning outcomes of our students.

4. CONCLUSIONS

In modern education, one of the most important goals is to enable students to understand graphic representations of natural and social phenomena, which is considered a necessary competence in professional and everyday communication.
Through the work with students, it was confirmed that the realization of the cognitive-visual approach in a computer environment is possible and, moreover, that it is recommended to be used as often as possible due to the multiple advantages it has in relation to traditional methods and procedures in teaching practice. Whenever it is possible for certain mathematical contents to receive a visual interpretation, then this possibility should be adequately used. In this sense, the continuous application of the cognitive-visual approach is recommended, with well-designed contents that will enable this methodical approach to produce the best effects, as well as with a combination with other effective teaching methods and procedures.
Contents, for which it is established that the cognitive-visual approach to their study gives positive effects, should be adequately processed methodically in textbooks, in order to become a guide for teachers and students on how to apply cognitive-visual didactic instructions and learning strategies.
Longitudinal monitoring of the effects of the application of the cognitive-visual approach in a computer environment on student achievement would significantly contribute to the optimization of methods, strategies, and approaches to the implementation of mathematics teaching in the future.

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