SOLVING INITIAL-VALUE PROBLEM BY COMPARING LAPLACE TRANSFORM AND AN NUMERICAL PREDICTOR-CORRECTOR METHOD

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Abstract: Different analytic and numerical methods have been proposed to solve differential equations. The goal of this paper is to expose the reader to modern computational tools for solving differential initial-value real problems of the form $y' = f(x, y)$; $y(x_0) = y_0$. The flow of electromagnetic forces, the temperature of a cup of coffee, the population of species as well as numerous other applications can be expressed as first order differential equations. Process of modeling or formulating a mathematical model of a real-world problem through either: intuitive reasoning about the phenomenon or a physical law based on evidence from experiments and often takes the form of a differential initial-value problem. It is noticed that changes occur, and we want to predict future behavior on the basis of how current values changes, and the model of growth population is based on the assumption that the population grows at the rate proportional to the size of the population. Many differential initial-value problems cannot be solved using analysis and because of that a numeric approximation to the solution is often sufficient. The algorithm of FABM method can be used to compute such an approximation of the problem. The idea is to compare the solutions of the taken problem, analytically and numerically using Adams-Bashforth-Moulton (FABM) like a predictor-corrector numerical method with solutions obtained by Laplace Transform (LT) like an analytical method. Laplace Transform like a powerful tool in applied mathematics and engineering, can be especially useful when dealing with piecewise-defined forcing functions. A predictor-corrector method is a set of two equations for y_{n+1}. The first equation, called the predictor, is used to predict (obtain a first approximation) to y_{n+1}, the second equation, called the corrector, is then used to obtain a corrected value (second approximation) to y_{n+1}, it depends on the predicted value. Approaches from FABM are based on the analytical property that the initial value problem is equivalent to the Volterra integral equation, in the sense that a continuous function is a solution of the initial value problem if and only if it is a solution of it. For calculations we chose a population problem with fixed initial condition and growth rate with changing the rate at which people are being added to or subtracted from the population during the immigration and emigration. The results are shown by time series solution, tables and phase portraits of the model taken by algorithms coded on Mathematica Package. Phase portraits serve as a useful tool in understanding the behavior of solutions of the problem, depending on variable rate depended on immigration and emigration of the population. Illustrative example can be included to demonstrate the validity and applicability of the present techniques.
Keywords: differential initial-value problem, population problem, Laplace transform, Adams-Bashforth-Moulton method, numerical analysis.

1. INTRODUCTION
Differential equations are among the linchpins of modern mathematics which, along with matrices, are essential for analyzing and solving complex real problems in engineering, the natural sciences, economics, business etc.
The basic theory of ordinary differential equations (ODEs) and analytical methods for solving some types of ODEs are reviewed. The differential equations are applied to all kinds of models, but this chapter concentrates on chemical kinetics and electric circuits in particular. A differential equation is an equation involving a relation between an unknown function and one or more of its derivatives. Equations involving derivatives of only one independent variable are called ordinary differential equations and may be classified as either initial-value problems or boundary value problems. (Abell & Braselton, 2004)
Mathematical models using differential equations consist to real-life situations, solved using technology and discussed or interpreted in light to understand the problem useful in the future.
A differential equation is an equation involving an unknown function and its derivatives. A solution of a differential equation in the unknown function y and the independent variable t on the interval I, is a function $y(t)$ that satisfies the differential equation identically for all t in I . (Abell & Braselton, 1994; Lynch, 2017)

Definition 1.1. The standard form for a first-order differential equation in the unknown function $y(t)$ is $y' = f(y, t)$. (Bronson & Costa, 2014)

Definition 1.2. A differential equation along with subsidiary conditions on the unknown function and its derivatives, all given at the same value of the independent variable, constitutes an initial-value problem of the form:

$$y' = f(y, t), \ y(t_0) = t_0 \tag{1}$$

(Bronson & Costa, 2014)

2. LAPLACE TRANSFORM

Definition 2.1. Let $f(t)$ be defined for $t \in [0, \infty)$ and let s denote an arbitrary real variable. The Laplace transform of $f(t)$ designated by either $L[f(t)]$ or $F(s)$, is:

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t) dt \tag{2}$$

for all values of s for which the improper integral converges. Convergence occurs when the limit $\lim_{R \to \infty} \int_0^R e^{-st} f(t) dt$, exists. If this limit does not exist, the improper integral diverges and $f(t)$ has no Laplace transform. (Bronson & Costa, 2014)

Properties of Laplace Transform

1. If $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$ then $L[c_1 f(t) + c_2 g(t)] = c_1 F(s) + c_2 G(s)$, for c_1, c_2 any constants.
2. $L[e^{at} f(t)] = F[s - a]$, for any constant a.
3. $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$.
4. If, for $t > 0$, $\lim_{t \to 0} \frac{f(t)}{t}$ exist, then $L[\frac{f(t)}{t}] = \int_s^\infty F(\lambda) d\lambda$.
5. $L\left[\int_0^t f(\lambda) d\lambda\right] = \frac{F(s)}{s}$.
6. If $f(t)$ is periodic, with period ω, then $L[ft] = \frac{\int_0^\omega e^{-st} f(t) dt}{1 - e^{-\omega t}}$.

Inverse Laplace Transform

Definition 2.2. The inverse Laplace transform of the function $F(s)$ is the unique continuous function $f(t)$ on $[0, \infty)$ that satisfies $L[f(t)] = F(s)$. We denote the inverse Laplace transform of $F(s)$ as

$$f(t) = L^{-1}[F(s)] \tag{3}$$

If the only functions that satisfy this relationship are discontinuous on $[0, \infty)$, we choose a piecewise continuous function on $[0, \infty)$ to be $L^{-1}[F(s)]$. Laplace transforms can be used to solve certain initial-value problems. (Bronson & Costa, 2014)

Typically, when we use it to solve that kind of problem for a function y, we do the following steps:

1. Compute the Laplace transform of each term in the differential equation.
2. Solve the resulting equation for $L[y(t)]$.
3. Determine y by computing the inverse Laplace transform of $L[y(t)]$. By using Laplace transform of derivation $L[f^n(t)] = s^n F(s) - s^{n-1} f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$ we transform a linear differential equation to an algebraic equation.

3. ADAMS-BASHFORTH-MOULTON METHOD

Methods using the approximation at more than one previous mesh point to determine the approximation at the next point are called multistep methods. A predictor-corrector method is a set of two equations for y_{n+1}.

The first equation, called the predictor, is used to predict y_{n+1}; the second equation, called the corrector, is then used to obtain a corrected value to y_{n+1}. In general, the corrector depends on the predicted value. Consider the following fourth-order method for solving an initial-value problem. The first step is to calculate the starting values $\omega_0, \omega_1, \omega_2, \omega_3$ for the four-step explicit Adams-Bashforth method.

To do this, we use a fourth-order one-step method, the Runge-Kutta method of order four. The next step is to calculate an approximation, ω_{4p} to $y(t_4)$ using the explicit Adams-Bashforth method as predictor:

$$\omega_{4p} = \omega_3 + \frac{h}{24}[55f(t_3, \omega_3) - 59f(t_2, \omega_2) + 37f(t_1, \omega_1) - 9f(t_0, \omega_0)] \tag{4}$$

Three-step implicit Adams-Moulton method, gives:

$$\omega_4 = \omega_3 + \frac{h}{24}[9f(t_4, \omega_{4p}) + 19f(t_3, \omega_3) - 5f(t_2, \omega_2) + f(t_1, \omega_1)] \tag{5}$$

The only new function evaluation required in this procedure is $f(t_4, \omega_{4p})$ in the corrector equation, all the other values of f have been calculated for earlier approximations.

The combination of an explicit method used to predict the value and an implicit method used to correct the value is called the predictor-corrector method. If the order of the predictor method is at least equal to the order of the corrective method, then a single iteration is sufficient to maintain the accuracy of the corrective method.

The most common predictor-corrector method is the combination of the formula of Adams-Bashforth as a predictor, with the formula of the fourth order of Adams Moulton as a corrector:

$$\begin{cases} \omega_{4p} = \omega_3 + \dfrac{h}{24}[55f(t_3, \omega_3) - 59f(t_2, \omega_2) + 37f(t_1, \omega_1) - 9f(t_0, \omega_0) \\ \omega_4 = \omega_3 + \dfrac{h}{24}[9f(t_4, \omega_{4p}) + 19f(t_3, \omega_3) - 5f(t_2, \omega_2) + f(t_1, \omega_1) \end{cases}$$

This approximation is improved by inserting ω_{4p} in the right side of the three-step implicit Adams-Moulton method is used like a corrector of the values taken by the explicit fourth -order Adams-Bashforth method as predictor.

4. APPLICATION TO POPULATION PROBLEM

Let $y(t)$ represent the population of a certain country. The rate at which the population increases and decreases depends on the growth rate of the country as well as the rate at which people are being added to or subtracted from the population due to immigration or emigration. (Mathews, 2013)

Hence, we consider the population problem:

$$y'(t) + ky = 1000(1 + a\sin t), \quad y(0) = y_0 \tag{6}$$

We will solve the problem for $k = 3, y_0 = 2000$ and varied $a = 0.2$, $a = 0.8$, firstly, by using Laplace Transform like an analytical method which obtains exact solutions and then by Adams-Bashforth Moulton method, like an multistep numerical method, which obtains approximated solutions, to compare the results geometrically and analytically. Results will be taken by using Mathematica Package for the both methods.

Laplace transform output of the problem is:

$$3 \, \text{LaplaceTransform}[x[t], t, s] + s \, \text{LaplaceTransform}[x[t], t, s] - x[0] == 1000 \left(\frac{1}{s} + \frac{a}{1+s^2} \right)$$

The result for given initial condition $y_0 = 2000$:

$$-2000 + 3 \, \text{LaplaceTransform}[x[t], t, s] + s \, \text{LaplaceTransform}[x[t], t, s] == 1000 \left(\frac{1}{s} + \frac{a}{1+s^2} \right)$$

$$\left\{ \left\{ \text{LaplaceTransform}[x[t], t, s] \rightarrow \frac{1000 \, (1 + 2 \, s + a \, s + s^2 + 2 \, s^3)}{s \, (3+s) \, (1+s^2)} \right\} \right\}$$

Inverse Laplace Transform of the result obtained by Mathematica Package:

$$\frac{100}{3} \left(10 + (50 + 3 \, a) \, e^{-3t} - 3 \, a \, (\text{Cos}[t] - 3 \, \text{Sin}[t]) \right)$$

We use the results to investigate the population for the values of for $a = 0.2$ and $= 0.4$, the results obtained are shown on Figure 1, time-series plotted with blue and orange solid lines and time interval $t \in [0, 20]$.

Figure 1. Time-series exact solution of y(t) using Laplace Transform, for a = 0.2 (blue solid-line) and a = 0.4 (orange solid-line), initial value y_0 = 2000 and time interval $t \in [0, 20]$
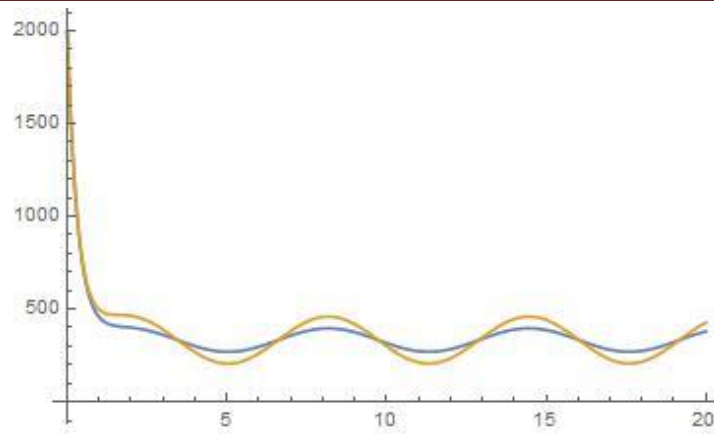
Figure 2. Time-series exact solution of $y(t)$ using Adams-Bashforth-Moulton, for $a = 0.2$ (blue solid-line) and $a = 0.4$ (orange solid-line), initial value $y_0 = 2000$ and time interval $t \in [0, 20]$
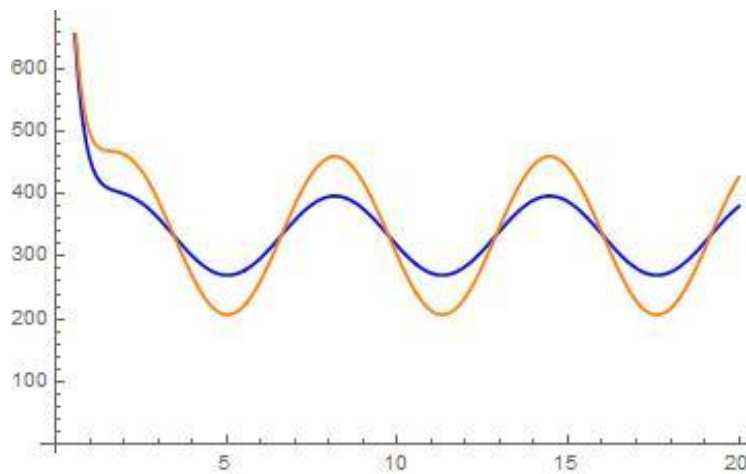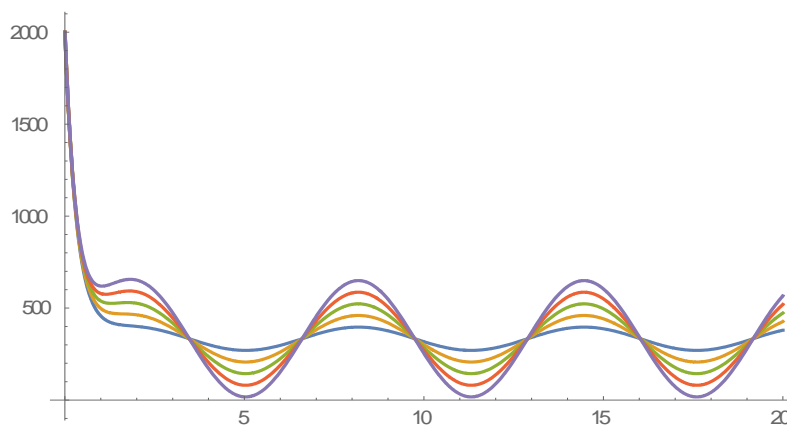

Figure 3. Fluctuations in the size of population using Laplace Transform method, for $a = 0.2$ (blue solid-line), $a = 0.4$ (orange solid-line), $a = 0.6$ (green solid-line), $a = 0.8$ (red solid-line) and $a = 0.4$ (purple solid-line).

For $a = 0.2$, to compare the methods , the results are shown in the Table 1. taken by Laplace Transform method and named like exact solution, with the ones taken by Adams-Bashforth-Moulton method named as approximated one. The result are in good agreement with each other.

Table 1. Solutions of equation with initial value $y_0 = 2000$, time interval $t \in [0, 0.2]$ with time space $h = 0.01$, LT method and FABM method

t	y_{exact}	y_{FABM}
0.01	1950.75	1945
0.02	1902.97	1892.39
0.03	1856.63	1846.34
0.04	1811.68	1801.68
0.05	1768.08	1758.36
0.06	1725.78	1761.34
0.07	1684.76	1675.58
0.08	1644.97	1635.05
0.09	1606.37	1597.7
0.1	1568.93	1560.51
1.02	1532.62	1523.43
1.02	1497.4	1489.44
1.03	1463.24	1455.5
1.04	1430.12	1422.59
1.05	1397.99	1390.71
1.06	1366.83	1359.71
1.07	1336.6	1329.68
1.08	1307.3	1300.56
1.09	1278.87	1272.33
0.2	1251.31	1244.94

5. CONCLUSIONS

Laplace transform can be used to solve varied applications of first-order equations and systems, especially an population problem. There are several problems of the field, logistic equation or Malthus model which can be investigated using the methods, LT or FABM. Almost everything in our world grows and decays, from the growth of organism to their eventual death, from growth of suns to their eventual supernovas, from growth of crystals to the decay of radioactive isotopes, from the growth of animal populations to the decay of radio signals etc.

From the time-series solution of the problem, depended of parameter a, we can see an decay of the population during the time, first month the decay is more pronounced till the first year, then we have an periodic display of the function. Similar results are obtained with two proposed algorithms, illustrated results on the Figure 1, Figure 2 and Table 1. Fluctuations in the size of population are larger for larger values of a, compared $a = 0.2$, $a = 0.4$, $a = 0.6$, $a = 0.8$ and $a = 1.0$, illustrated on the Figure 3, and calculated using LT method.

REFERENCES

Abell, M.L., & Braselton, J.P. (2004). Differential Equations with Mathematica. USA. Elsevier INC.
Abell, M.L., & Braselton, J.P. (1994). Mathematica by example. USA. Academic Press, Inc.
Atici, F.M., & Senguel, S. (2010). Modeling with fractional difference equations. *J. Math. Anal. Appl*. 369, 1–9.
Bronson, R., & Costa, G. B. (2014). Differential Equations. Schaums Outline's. USA. McGraw-Hill Education.
Holm, M.T. (2011).The Laplace transform in discrete fractional calculus. *Comput. Math. Appl*. 62, 1591–1601.
Khennaoui, A.A., Ouannas, A., Bendoukha, S., Grassi, G., Lozi, R.P., & Pham, V.T. (2019). On fractional–order discrete–time systems: Chaos, stabilization and synchronization. *Chaos Solitons Fractals*, *119*, 150–162.
Luo, H. (2007). Population Modeling by Differential Equations. Theses, Dissertations and Capstones. Marshall University.
Lynch, S. (2017). Dynamical Systems with Applications using Mathematica. Manchester, United Kingdom. Manchester Metropolitan University.

Martelli, M. (1999). Introduction to discrete dynamical systems and chaos. Canada. John Wiley & Sons, INC.
Mathews, D. (2013). Growth and decay – A guide for teachers (Years 11–12). Australia. Education Services Australia.
May, R.M. (1976). Simple mathematical models with very complicated dynamics. *Nature* 261, 459–467.
Zill, D.D. (2013). A First Course in Differential Equations with Modeling Applications. USA. Richard Stratton.